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COUPLING REQUIREMENTS FOR WELL POSED AND STABLE MULTI-PHYSICS PROBLEMS

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Abstract. We discuss well-posedness and stability of multi-physics problems by studying a model problem. By applying the energy method, boundary and interface conditions are derived such that the continuous and semi-discrete problem are well-posed and stable. The numerical scheme is implemented using high order finite difference operators on summation-by-parts (SBP) form and weakly imposed boundary and interface conditions. Numerical experiments involving a spectral analysis corroborate the theoretical findings.

1 INTRODUCTION

Roughly speaking, a well posed initial boundary value problem require that a unique solution that can be estimated in terms of the data, exist. The most common procedure for showing well posedness is the so called energy-method where one multiply the governing partial differential equations (PDEs) with the solution, integrate by parts and impose boundary conditions [1]. The same general knowledge is not wide-spread when it comes to the mathematical coupling of multi-physics problems. The reason for that is the more complex and to some extent more unclear nature of coupling conditions compared to imposing boundary conditions.

Firstly, accuracy relations must exist such that combinations of variables for one set of PDEs at the interface is equal to combinations of variables for the other set. Secondly, the number of accuracy relations must fit both problems. Too many conditions ruin existence and too few ruin uniqueness. If the number of accuracy relations are too few, additional conditions requiring external data must be added. Thirdly, the accuracy relations must be such that no artificial growth or decay is generated.

We will investigate the problems mentioned above and generalize the investigation in [2, 3] where we derived the coupling conditions by only demanding a well posed problem. Coupling of hyperbolic PDEs of different size at the interface will be our primary focus. Once the coupling conditions are known for the continuous multi-physics problem we will discretize using high order finite differences on summation-by-parts form and include the coupling conditions weakly using the SAT technique [4, 5].

2 THE MODEL PROBLEM

We will consider the following system,

$$\begin{aligned} u_t + Au_x &= 0, \qquad -1 \le x \le 0, \qquad t > 0, \\ u(x,0) &= f(x), \end{aligned} \tag{1}$$

and the scalar equation

$$\begin{aligned} v_t + bv_x &= 0, \qquad 0 \le x \le 1, \qquad t > 0, \\ v(x,0) &= g(x). \end{aligned} \tag{2}$$

In (1), $u = (u_1, u_2)^T$ is a vector of unknowns, $f(x) = (f_1(x), f_2(x))^T$ is a vector of given data and for simplicity we choose

$$A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}, a > 0. \tag{3}$$

Two boundary/interface conditions are needed for the system (1) while equation (2) needs one boundary/interface condition.

2.1 The interface conditions

We apply the energy method to both equations and add them together to get

$$\frac{d}{dt}(\|u\|_2^2 + \alpha\|v\|_2^2) = -u^T Au|_{x=0} + \alpha bv^2|_{x=0} = w^T Ew, \tag{4}$$

where α is positive free weight, $w = [u_1, u_2, v]^T$ and E is

$$E = \begin{bmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ 0 & 0 & \alpha b \end{bmatrix}. \tag{5}$$

In (4), the boundary terms at the outer boundaries $x = \pm 1$ are ignored. The eigenvalues of E are $\{a, -a, \alpha b\}$. If $b < 0$, one of the eigenvalues is positive and we need one condition at $x = 0$, otherwise we need two conditions.

In order to couple the problems we need at least one accuracy condition. Let

$$v = C^T u, \quad C = [c_1, c_2]^T. \tag{6}$$

The relation (6) inserted in equation (4) leads to

$$\frac{d}{dt}(\|u\|_2^2 + \alpha \|v\|_2^2) = u^T(0,t) D u(0,t), \tag{7}$$

where $D = (\alpha b C C^T - A)$. The characteristic polynomial related to the eigenvalues λ of D is

$$\lambda^2 - \alpha b(c_1^2 + c_2^2)\lambda + 2\alpha a b c_1 c_2 - a^2. \tag{8}$$

To simplify the following discussion we let $2s_1 = -\alpha b(c_1^2 + c_2^2)$ and $s_2 = 2\alpha a b c_1 c_2 - a^2$, which yields the roots

$$\lambda_{1,2} = -s_1 \pm \sqrt{s_1^2 - s_2}. \tag{9}$$

First we consider $b < 0$. This leads to a positive s_1. If $c_1 c_2 \leq a/2b\alpha$, then $s_2 \geq 0$ and both roots of the characteristic polynomial are negative, which means that D is negative definite. This means that if c_1 and c_2 have opposite sign, the coupled problems satisfy an energy estimate for all choices of α. But if c_1 and c_2 have the same sign, the energy estimate is not satisfied for any value of α. Consequently the coupled problems with the interface condition $v = C^T u$ satisfy an energy estimate for $b < 0$ if and only if c_1 and c_2 have opposite signs.

Next, consider $b > 0$. This leads to negative s_1 and at least one of the eigenvalues must be positive, which means that we need an additional condition. As mentioned above, two conditions are needed at $x = 0$. One of them is an interface condition and the other one must be such that the right-hand side of (7) is negative semi-definite. We will refer to this additional condition as a boundary condition. If $c_1 c_2 \leq a/2b\alpha$, then $s_2 \leq 0$ and one of the eigenvalues of D is positive (λ^+) and the other one is negative (λ^-). Let $D = Y \Lambda Y^T$ and rewrite (7) as

$$\frac{d}{dt}(\|u\|_2^2 + \alpha \|v\|_2^2) = u^T(0,t)(Y \Lambda Y^T) u(0,t), \tag{10}$$

where $\Lambda = \mathrm{diag}\{\lambda^+, \lambda^-\}$ and Y is the matrix of eigenvectors to D. Let $\Lambda = \Lambda^+ + \Lambda^-$, where $\Lambda^+ = \mathrm{diag}\{\lambda^+, 0\}$ and $\Lambda^- = \mathrm{diag}\{0, \lambda^-\}$. Furthermore we have $D = D^+ + D^-$ where $D^+ = Y \Lambda^+ Y^T$ and $D^- = Y \Lambda^- Y^T$. Then (10) leads to

$$\frac{d}{dt}(\|u\|_2^2 + \alpha \|v\|_2^2) = (Y^T u(0,t))^T (\Lambda^+ + \Lambda^-)(Y^T u(0,t)), \tag{11}$$

The most general condition based on (11) is

$$(Y_+^T - R_r Y_-^T)u(0,t) = h(t), \qquad x = 0, \tag{12}$$

where Y_+ and Y_- are the eigenvectors related to the positive and negative eigenvalues, respectively. Letting $h(t) = 0$ and inserting (12) into (11) leads to

$$\frac{d}{dt}(\|u\|_2^2 + \alpha\|v\|_2^2) = (\lambda^- + R_r^2\lambda^+)(Y_-^T u(0,t))^2. \tag{13}$$

If $\lambda^- + R_r^2\lambda^+ \leq 0$, then the right-hand side of (13) is bounded and we have a well-posed coupling. Note that with $R_r = 0$ we have the so called characteristic boundary conditions.

Consequently an energy estimate is obtained if c_1 and c_2 are chosen such that $c_1 c_2 \leq a/2b\alpha$. This means that c_1 and c_2 must be less than an arbitrary positive number that we can choose. In short: all values of c_1 and c_2 lead to a well-posed problem if $b > 0$.

2.2 The semi-discrete problem

Let A be an $M \times N$ matrix and B a $P \times R$ matrix. The Kronecker product of these matrices is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1N}B \\ \cdots & & \cdots \\ a_{M1}B & \cdots & a_{MN}B \end{bmatrix}. \tag{14}$$

First, we consider $b < 0$. The semi-discrete SBP-SAT formulations of (1) and (2) are,

$$\begin{aligned} \mathbf{u}_t + (D_u \otimes A)\mathbf{u} &= (P_u^{-1}E_N^u \otimes \Sigma)(C^T\tilde{u}_N - v_0)e_N^u, \\ \mathbf{v}_t + bD_v\mathbf{v} &= P_v^{-1}\sigma(v_0 - C^T\tilde{u}_N)e_0^v. \end{aligned} \tag{15}$$

In (15), the outer boundary conditions are ignored as in the continuous case, $D_{u,v} = P_{u,v}^{-1}Q_{u,v}$ are the difference operators, $P_{u,v}$ are positive definite matrices and $Q_{u,v}$ satisfy $Q_{u,v} + Q_{u,v}^T = diag[-1, \cdots, 1]$. The discrete grid functions, related to the grid vectors $x_u = (x_0 = -1, \cdots, x_N = 0)$ and $x_v = (y_0 = 0, \cdots, y_M = 1)$ are

$$\mathbf{u} = (u_{10}, u_{20}, \cdots u_{1N}, u_{2N}), \qquad \mathbf{v} = (v_0, \cdots, v_M). \tag{16}$$

The vectors $e_N^u = (0, \cdots, 0, 1, 1)^T$ and $e_0^v = (1, \cdots, 0)^T$ are $2N \times 1$ and $M \times 1$, respectively. $E_N^u = diag[0, \cdots, 1]$ and $E_0^v = diag[1, \cdots, 0]$ are $N \times N$ and $M \times M$, respectively. The penalty matrix Σ is given by

$$\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{bmatrix}, \tag{17}$$

σ is penalty parameter and $\tilde{u}_N = [u_{1N}, u_{2N}]^T$.

Next, consider $b > 0$. The semi-discrete SBP-SAT formulations of (1) and (2) are,

$$\mathbf{u}_t + (D_u \otimes A)\mathbf{u} = (P_u^{-1}E_N^u \otimes \Sigma)(C^T\tilde{u}_N - v_0)e_N^u + (P_u^{-1}E_N^u \otimes \Xi)((I_N \otimes \tilde{H})\mathbf{u} - e_N^u \otimes \tilde{h}),$$
$$\mathbf{v}_t + bD_v\mathbf{v} = P_v^{-1}\sigma(v_0 - C^T\tilde{u}_N)e_0^v. \tag{18}$$

where the penalty matrix Ξ and $\tilde{H}$ are given by

$$\Xi = \begin{bmatrix} \chi_1 & \chi_2 \\ \chi_3 & \chi_4 \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} 1 & -R_r \\ 0 & 0 \end{bmatrix} Y^T. \tag{19}$$

The boundary data $\tilde{h}$ is defined as $\tilde{h} = [0, h]^T$. Also, in the following analysis we will use the discrete norms

$$\|\mathbf{u}\|^2_{P_u \otimes I} = \mathbf{u}^T(P_u \otimes I)\mathbf{u}, \qquad \|\mathbf{v}\|^2_{P_v} = \mathbf{v}^T P_v \mathbf{v}. \tag{20}$$

2.2.1 Stability conditions at the interface

First we consider $b < 0$. The discrete energy method is applied to (15) by multiplying the two equations with $\mathbf{u}^T(P_u \otimes I)$ and $\mathbf{v}^T P_v$, respectively. The SBP properties of $D_{u,v}$ yields

$$\frac{d}{dt}(\|\mathbf{u}\|^2_{P_u\otimes I} + \alpha_d\|\mathbf{v}\|^2_{P_v}) = -\tilde{u}_N^T A\tilde{u}_N + \alpha_d b v_0^2 + 2\tilde{u}_N^T\Sigma H + 2\alpha_d\sigma v_0(v_0 - C^T\tilde{u}_N). \tag{21}$$

In(21), α_d is a positive weight (not necessarily the same as in the continuous case) and $H = [C^T\tilde{u}_N - v_0, C^T\tilde{u}_N - v_0]^T$. In order to mimic the continuous case, we choose $\sigma_2 = c_1\alpha b/2$ and $\sigma_4 = c_2\alpha b/2$ and $\sigma_1 = \sigma_3 = 0$. The final penalty matrix in block form becomes $\Sigma = \alpha b/2 \begin{bmatrix} 0 & C \end{bmatrix}$. By inserting that into (21) we get

$$\frac{d}{dt}(\|\mathbf{u}\|^2_{P_u\otimes I} + \alpha_d\|\mathbf{v}\|^2_{P_v}) = \tilde{u}_N^T D\tilde{u}_N + \alpha_d v_0^2(b + 2\sigma) - \sigma v_0 C^T\tilde{u}_N(\alpha b + 2\alpha_d\sigma). \tag{22}$$

If we choose $\sigma = -\alpha b/2\alpha_d$, for $\alpha_d \leq \alpha$ the right-hand side of (22) will be bounded due to the continuous result above.

Next, we consider $b > 0$ and let $h(t) = 0$. Multiplying (18) by $\mathbf{u}^T(P_u \otimes I)$ and $\mathbf{v}^T P_v$ leads to

$$\frac{d}{dt}(\|\mathbf{u}\|^2_{P_u\otimes I} + \alpha_d\|\mathbf{v}\|^2_{P_v}) \leq \tilde{u}_N^T(D + \Xi\tilde{H} + (\Xi\tilde{H})^T)\tilde{u}_N, \tag{23}$$

where we have chosen Σ and σ as for the case $b < 0$. By using $YY^T = I$, we can rewrite the right-hand side of (23) as

$$\tilde{u}_N^T(D + \Xi\tilde{H} + (\Xi\tilde{H})^T)\tilde{u}_N = (Y^T\tilde{u}_N)^T(\Lambda + (Y^T\Xi\tilde{H}Y) + (Y^T\Xi\tilde{H}Y)^T)(Y^T\tilde{u}_N). \tag{24}$$

Let $\tilde{\Xi} = Y^T \Xi$ and choose Ξ such that $\tilde{\Xi} = diag(\tilde{\chi}_1, \tilde{\chi}_2)$. We also use the following split,

$$(Y^T \tilde{u}_N) = \begin{bmatrix} (Y_+^T \tilde{u}_N) \\ (Y_-^T \tilde{u}_N) \end{bmatrix}. \tag{25}$$

Now, we can rewrite (24) as,

$$\tilde{u}_N^T (D + \Xi \tilde{H} + (\Xi \tilde{H})^T) \tilde{u}_N = \begin{bmatrix} (Y_+^T \tilde{u}_N) \\ (Y_-^T \tilde{u}_N) \end{bmatrix}^T \begin{bmatrix} \lambda^+ + 2\tilde{\chi}_1 & -R_r \tilde{\chi}_1 \\ -R_r \tilde{\chi}_1 & \lambda^- \end{bmatrix} \begin{bmatrix} (Y_+^T \tilde{u}_N) \\ (Y_-^T \tilde{u}_N) \end{bmatrix}. \tag{26}$$

By the choice $\tilde{\chi}_1 = -\lambda^+$, the right-hand side of (26) can be rewritten as

$$(\lambda^- + R_r^2 \lambda^+)(Y_-^T \tilde{u}_N)^2 - \lambda^+((Y_+^T \tilde{u}_N) - R_r(Y_-^T \tilde{u}_N))^2, \tag{27}$$

which is negative due to the continuous result. Consequently if we choose

$$\Sigma = \alpha b/2 \begin{bmatrix} 0 & C \end{bmatrix}, \; \sigma = -\alpha b / 2\alpha_d, \; \Xi = Y \begin{bmatrix} -\lambda^+ & 0 \\ 0 & 0 \end{bmatrix},$$

then for $\alpha_d \geq \alpha$, (18) is stable.

2.2.2 Stability conditions at the left boundary

In order to have a well-posed problem, we need one condition at $x = -1$. We consider the homogeneous boundary condition

$$(X_+^T - R_l X_-^T) u(-1, t) = 0, \tag{28}$$

with $|R_l| < 1$. The SAT term at $x = -1$ is $(P_u^{-1} E_0^u \otimes \Pi)(I_N \otimes \hat{H})\mathbf{u}$, where

$$\Pi = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_3 & \pi_4 \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} 1 & -R_l \\ 0 & 0 \end{bmatrix} X^T, \tag{29}$$

and X is the matrix of eigenvectors to A. It can be shown that an energy estimate is obtained if

$$\pi_1 = -a/(4(R_l + 1)), \qquad \pi_2 = 0, \qquad \pi_3 = a/(4(R_l - 1)), \qquad \pi_4 = 0. \tag{30}$$

2.2.3 Stability conditions at the right boundary

For the case $b < 0$, one condition at $x = 1$ is also needed. We choose the homogeneuos $v(1, t) = 0$. The SAT term at $x = 1$ is $P_v^{-1} \theta v_N^2 e_N^v$ where θ satisfies

$$\theta \leq b/2. \tag{31}$$

3 THE SPECTRUM

In this section, we consider the continuous and discrete spectrum for our problem.

3.1 The spectrum for the continuous problem

By applying the Laplace transform to (1) and (2) we get the following system of ordinary differential equations

$$\begin{aligned} s\hat{u} + A\hat{u}_x &= 0, \quad -1 \leq x \leq 0, \\ s\hat{v} + b\hat{v}_x &= 0, \quad 0 \leq x \leq 1. \end{aligned} \tag{32}$$

We have ignored the initial conditions, since they do not influence the spectra and make the ansatz $\hat{u} = e^{kx}\psi$ and $\hat{v} = e^{k_3 x}\psi_3$. This leads to

$$(sI + Bk)\Psi = 0, \qquad B = \begin{bmatrix} A & 0 \\ 0 & b \end{bmatrix}, \tag{33}$$

where $\Psi = [\psi_1, \psi_2, \psi_3]^T$. This system of equations have a non-trivial solution only when $\det(sI + Bk) = 0$, which leads to $k_1 = -\frac{s}{a}, k_2 = \frac{s}{a}$ and $k_3 = -\frac{s}{b}$.

The general solution including the eigenvectors is

$$\hat{w} = \alpha_1 e^{-\frac{s}{a}x} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 e^{\frac{s}{a}x} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \alpha_3 e^{-\frac{s}{b}x} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{34}$$

where $\hat{w} = [\hat{u}_1, \hat{u}_2, \hat{v}]^T$. The unknowns α_1, α_2 and α_3, will be determined by the boundary and interface conditions.

First we consider $b < 0$, with the conditions

$$\begin{aligned} (X_+^T - R_l X_-^T)u(-1,t) &= 0, \\ C^T u(0,t) - v(0,t) &= 0, \\ v(1,t) &= 0, \end{aligned} \tag{35}$$

and $|R_l| \leq 1$. The interface and boundary conditions are such that the coupled problem is well-posed. By applying these conditions to (34), we obtain

$$E\alpha = 0, \qquad E = \begin{bmatrix} c_1 + c_2 & c_1 - c_2 & -1 \\ 2e^{\frac{s}{a}} & -2R_l e^{-\frac{s}{a}} & 0 \\ 0 & 0 & e^{-\frac{s}{b}} \end{bmatrix}, \tag{36}$$

where $\alpha = [\alpha_1, \alpha_2, \alpha_3]^T$. A non-trivial solution, require

$$\det(E) = 2e^{-\frac{s}{b}}(e^{\frac{s}{a}}(c_2 - c_1) - R_l e^{-\frac{s}{a}}(c_1 + c_2)) = 0. \tag{37}$$

The zeros of $\det(E)$ which form the spectrum of (32) are

$$s = \begin{cases} \frac{a}{2}\ln(|\frac{R_l(c_1+c_2)}{c_2-c_1}|) + na\pi i, & n \in \mathbb{Z}, \ \text{if} \frac{R_l(c_1+c_2)}{c_2-c_1} > 0, \\ \frac{a}{2}\ln(|\frac{R_l(c_1+c_2)}{c_2-c_1}|) + na\pi i + \frac{a\pi i}{2}, & n \in \mathbb{Z}, \ \text{if} \frac{R_l(c_1+c_2)}{c_2-c_1} < 0. \end{cases} \tag{38}$$

The real part of s is negative if $|\frac{R_l(c_1+c_2)}{c_2-c_1}| < 1$. It is easy to verify that this holds for $|R_l| < 1$ and c_1, c_2 with opposite sign. This means that if we choose c_1, c_2 and R_l such that the coupled problem leads to an energy estimate, then the real part of s will be negative. Recall that this required that c_1 and c_2 must have opposite signs.

Next, consider $b > 0$. In this case we have the conditions

$$\begin{aligned} (X_+^T - R_l X_-^T)u(-1,t) &= 0, \\ C^T u(0,t) - v(0,t) &= 0, \\ (Y_+^T - R_r Y_-^T)u(0,t) &= 0, \end{aligned} \tag{39}$$

and $\lambda^- + R_r^2\lambda^+ \leq 0$ and $R_l^2 \leq 1$. The coupled problems with the conditions (39) satisfy an energy estimate. By applying (39) to (34) leads to the following system of equations

$$E\alpha = 0, \qquad E = \begin{bmatrix} c_1 + c_2 & c_1 - c_2 & -1 \\ 2e^{\frac{s}{a}} & -2R_l e^{-\frac{s}{a}} & 0 \\ y_{12} - R_r y_{11} + 1 - R_r & y_{12} - R_r y_{11} - 1 + R_r & 0 \end{bmatrix}, \tag{40}$$

where $\alpha = [\alpha_1, \alpha_2, \alpha_3]^T$. The zeros of $\det(E)$ in this case are

$$s = \begin{cases} \frac{a}{2}\ln(|\frac{R_l(y_{12}-R_r y_{11}+1-R_r)}{-y_{12}+R_r y_{11}+1-R_r}|) + na\pi i, & n \in \mathbb{Z}, \ \text{if} \frac{R_l(y_{12}-R_r y_{11}+1-R_r)}{-y_{12}+R_r y_{11}+1-R_r} > 0, \\ \frac{a}{2}\ln(|\frac{R_l(y_{12}-R_r y_{11}+1-R_r)}{-y_{12}+R_r y_{11}+1-R_r}|) + na\pi i + \frac{a\pi i}{2}, & n \in \mathbb{Z}, \ \text{if} \frac{R_l(y_{12}-R_r y_{11}+1-R_r)}{-y_{12}+R_r y_{11}+1-R_r} < 0. \end{cases} \tag{41}$$

The real part of s is negative if

$$|\frac{R_l(y_{12} - R_r y_{11} + 1 - R_r)}{-y_{12} + R_r y_{11} + 1 - R_r}| < 1. \tag{42}$$

Note that the determinant of E is independent of c_1 and c_2. This means that if (42) holds, then the real part of s is negative for all c_1 and c_2. Recall that for all values of c_1 and c_2, suitable choices of α such that the coupled problems satisfy an energy estimate could be found. This implies that there is no limitation on c_1 and c_2 in both the energy and spectral analysis. However, recall that this required an additional boundary condition.

3.2 The semi-discrete spectrum

Consider $b < 0$. The SBP-SAT approximation of (1) and (2), including (35) is

$$\begin{aligned} \mathbf{u}_t + (D_u \otimes A)\mathbf{u} &= (P_u^{-1}E_0^u \otimes \Pi)(I_N \otimes \hat{H})\mathbf{u} + (P_u^{-1}E_N^u \otimes \Sigma)(C^T\tilde{u}_N - v_0)e_N^u, \\ \mathbf{v}_t + bD_v\mathbf{v} &= P_v^{-1}\sigma(v_0 - C^T\tilde{u}_N)e_0^v + P_v^{-1}\theta v_N e_N. \end{aligned} \tag{43}$$

In order to determine the semi-discrete spectrum, we follow [6] and rewrite (43) in matrix form as

$$W_t = P^{-1}(H_i + H_c)W, \tag{44}$$

where $W = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}]^T$ and

$$H_i = \begin{bmatrix} -(Q_u \otimes A) & 0 \\ 0 & -bQ_v \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} P_u^{-1} \otimes I_2 & 0 \\ 0 & P_v^{-1} \end{bmatrix}. \tag{45}$$

The penalty matrix H_c which is zero except at the boundaries and interface has the structure

$$H_c = \begin{bmatrix} \Pi\hat{H} & & & & & \\ & \ddots & & & & \\ & & EC^T & -E & & \\ & & -\sigma C^T & \sigma & & \\ & & & & \ddots & \\ & & & & & \theta \end{bmatrix}, \tag{46}$$

where $E = [\sigma_2, \sigma_4]^T$.

Next, consider $b > 0$. The SBP-SAT approximation of (1) and (2), with conditions (39) is

$$\begin{aligned} \mathbf{u}_t + (D_u \otimes A)\mathbf{u} &= (P_u^{-1}E_0^u \otimes \Pi)(I_N \otimes \hat{H})\mathbf{u} + (P_u^{-1}E_N^u \otimes \Sigma)(C^T\tilde{u}_N - v_0)e_N^u \\ &\quad + (P_u^{-1}E_N^u \otimes \Xi)(I_N \otimes \tilde{H})\mathbf{u}, \\ \mathbf{v}_t + bD_v\mathbf{v} &= P_v^{-1}\sigma(v_0 - C^T\tilde{u}_N)e_0^v. \end{aligned} \tag{47}$$

The approximation (47) can be written on the the form (44) where in this case

$$H_c = \begin{bmatrix} \Pi\hat{H} & & & & & \\ & \ddots & & & & \\ & & EC^T + \Xi\tilde{H} & -E & & \\ & & -\sigma C^T & \sigma & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, \tag{48}$$

while H_i is the same as before and given above. The eigenvalues of the matrix $P^{-1}(H_i+H_c)$ form the discrete spectrum of (43) and (47).

4 NUMERICAL RESULTS

In this section, we use the method of manufactured solution in order to test the accuracy of the approximations. RK3 is used to discretize time. We also discuss the relation between the continuous and semi-discrete spectrum.

	$SBP21$		$SBP42$		$SBP63$		$SBP84$	
N	$error$	$rate$	$error$	$rate$	$error$	$rate$	$error$	$rate$
20	2e-2	-	2e-3	-	2e-3	-	1e-3	-
40	6e-3	1.877	3e-4	3.006	1e-4	4.035	3e-5	5.376
80	1e-3	2.046	3e-5	3.242	8e-6	4.224	8e-7	5.392
160	4e-4	1.985	3e-6	3.052	4e-7	4.470	2e-8	5.113
320	1e-4	2.004	4e-7	3.021	2e-8	4.375	6e-10	5.091
640	2e-5	1.998	6e-8	3.013	1e-9	4.077	2e-11	5.047

Table 1: error and rate q_u for $b < 0$.

	$SBP21$		$SBP42$		$SBP63$		$SBP84$	
N	$error$	$rate$	$error$	$rate$	$error$	$rate$	$error$	$rate$
20	2e-1	-	3e-2	-	3e-2	-	4e-3	-
40	4e-2	2.148	3e-3	3.154	1e-3	4.469	2e-4	4.288
80	1e-2	2.050	4e-4	3.046	6e-5	4.704	8e-6	4.741
160	2e-3	2.014	5e-5	3.011	2e-6	4.668	3e-7	4.916
320	6e-4	2.005	6e-6	3.003	1e-7	4.474	9e-9	4.798
640	2e-4	2.001	7e-7	3.002	4e-9	4.467	3e-10	4.832

Table 2: error and rate q_v for $b < 0$.

4.1 Accuracy

The analytical solution that we use in the method of manufactured solution is

$$u_1(x,t) = u_2(x,t) = \cos(2\pi(x-t)), v(x,t) = \sin(3\pi(x-bt)). \tag{49}$$

The rate of convergence is calculated as

$$q_u = \ln\left(\frac{\|(\mathbf{u}_1^{N_1}, \mathbf{u}_2^{N_1}) - (u_1,u_2)\|_{P_u\otimes I}}{\|(\mathbf{u}_1^{N_2}, \mathbf{u}_2^{N_2}) - (u_1,u_2)\|_{P_u\otimes I}}\right) / \ln\left(\frac{N_1}{N_2}\right), q_v = \ln\left(\frac{\|\mathbf{v}^{N_1} - v\|_{P_v}}{\|\mathbf{v}^{N_2} - v\|_{P_v}}\right) / \ln\left(\frac{N_1}{N_2}\right), \tag{50}$$

where u_1, u_2 and v are the analytical solutions and $u_1^{N_i}, u_2^{N_i}$ and v^{N_i} are the corresponding numerical solutions with N_i grid points.

First, we consider $b < 0$. The choosen coefficients are $\alpha = \alpha_d = 1, a = 1, b = -1$. To have a well-posed problem, we choose $|R_l| < 1$ and c_1, c_2 such that $c_1c_2 \leq -1/2$. Let $R_l = 0.25$ and $c_1 = 1, c_2 = -2$. Tables 1 and 2 show the error and convergence rate q_u and q_v, respectively, for SBP operators with $2^{th}, 3^{th}, 4^{th}$ and 5^{th} order. Next, we consider $b > 0$. We again choose $\alpha = \alpha_d = 1, a = 1, b = 1$ and take $R_l = 0.25, R_r = 0.25, c_1 = 1$ and $c_2 = 1$ in order to have a well-posed problem. Tables 3 and 4 show the error and convergence rates for q_u and q_v, respectively. Clearly, the design order of accuracy is obtained.

	$SBP21$		$SBP42$		$SBP63$		$SBP84$	
N	$error$	$rate$	$error$	$rate$	$error$	$rate$	$error$	$rate$
20	2e-2	-	5e-3	-	3e-3	-	2e-3	-
40	6e-3	1.885	6e-4	2.951	2e-4	4.186	1e-4	4.239
80	1e-3	2.062	8e-5	2.979	8e-6	4.367	3e-6	5.050
160	4e-4	1.983	1e-5	2.994	4e-7	4.492	8e-8	5.222
320	9e-5	2.005	1e-6	2.999	2e-8	4.392	2e-9	5.200
640	2e-5	1.998	1e-7	2.999	1e-9	4.321	5e-11	5.193

Table 3: error and rate q_u for $b > 0$.

	$SBP21$		$SBP42$		$SBP63$		$SBP84$	
N	$error$	$rate$	$error$	$rate$	$error$	$rate$	$error$	$rate$
20	3e-2	-	8e-3	-	7e-3	-	3e-3	-
40	7e-3	2.002	1e-3	3.083	4e-4	4.287	2e-4	4.427
80	1e-3	2.003	1e-4	2.954	1e-5	4.523	7e-6	4.542
160	4e-4	2.000	2e-5	2.983	7e-7	4.442	3e-7	4.773
320	1e-4	2.000	2e-6	2.989	3e-8	4.437	9e-9	4.753
640	3e-5	2.000	3e-7	2.994	2e-9	4.436	3e-10	4.749

Table 4: error and rate q_v for $b > 0$.

4.2 The spectrum of the continuous and semi-discrete operators

Figures 1-3 show the discrete and continuous spectrum for different grids using the SBP42 operator. One can clearly see the convergence of the discrete spectrum to the continuous one as the grids are refined. This convergence hold both for positive and negative b and show that the solutions of the semi-discrete scheme converge to the continuous one.

5 SUMMARY AND CONCLUSIONS

We have discussed well-posedness and stability of multi-physics problems by analyzing a model problem. It was shown that for ceartain wave speeds, only interface conditions were required, while in other cases additional information in the form of boundary conditions must be supplied.

By applying the energy method, we derived boundary and interface conditions such that the continuous and semi-discrete problem are well-posed and stable. The numerical scheme was implemented using high order finite difference operators on SBP form and weakly imposed boundary and interface conditions using the SAT technique.

It was shown that we obtained design order of accuracy, and that the spectrum of the

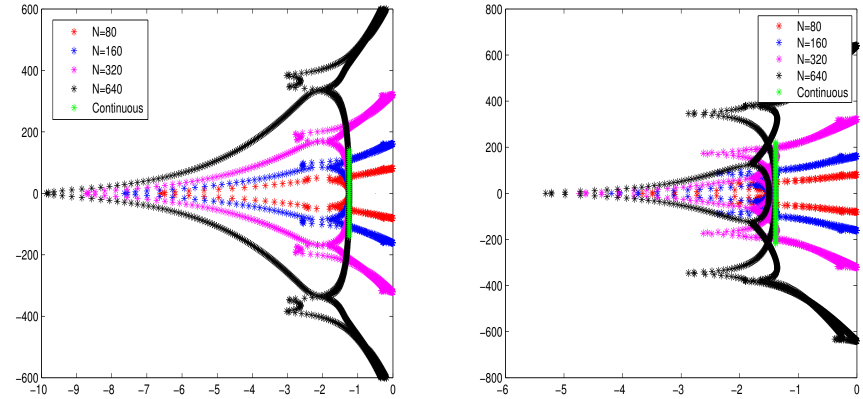

Figure 1: Global view: the discrete and continuous spectrum, $b < 0$ (left) and $b > 0$ (right).

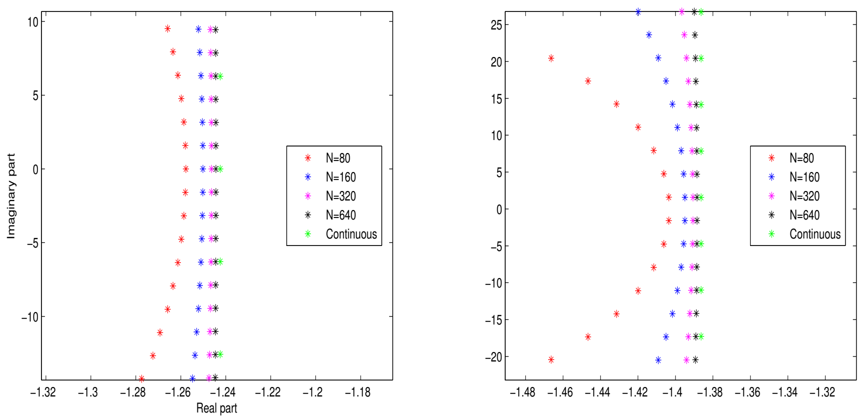

Figure 2: Medium view: the discrete and continuous spectrum, $b < 0$ (left) and $b > 0$ (right).

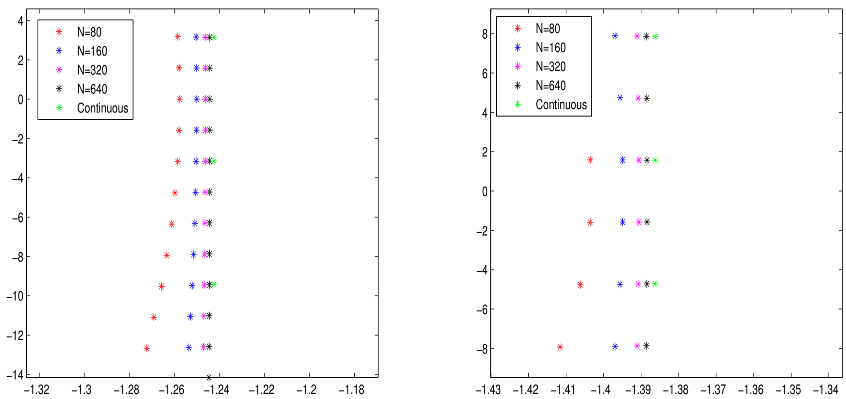

Figure 3: Zoomed view: the discrete and continuous spectrum, $b < 0$ (left) and $b > 0$ (right).

discrete operator converged to the spectrum of the continuous operator. The numerical experiments in combination with the theoretical derivations showed what type of analysis that is required to obtain accurate numerical simulations of multi-physics problems.

Future work will include a generalization of this investigation for hyperbolic problems, and an extension to coupling of incompletely parabolic problems such as the compressible Navier-Stoke's equations.

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